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BOUNDARY LAYER PHENOMENON IN THE PLASTIC ZONE NEAR A RAPIDLY PROPAGATING CRACK TIP

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Abstract—The steady-state dynamic fields near a rapidly propagating crack tip in an elastic perfectly-plastic material have been investigated for the case of Mode-III fracture. For arbitrary values of the dimensionless crack-tip speed (M) the inner solution consists of a central-fan field ahead of the crack tip and a uniform field in its wake. It is shown that the inner solution is valid in a "boundary layer" which shrinks on the crack tip in the limit of vanishing M. For small M the outer solution was found as a regular perturbation expansion in M, with the quasi-static solution as its first term. A uniform expansion over the polar angle θ measured from the plane of the crack was also obtained; its first term displays the connection between the inner and outer solutions. The expansion has also been used to obtain an exact solution for the shear strain in the plastic zone in the plane of the crack ahead of the crack tip.

1. INTRODUCTION

In a recent paper the authors have investigated dynamic effects on the fields of stress and strain near a rapidly propagating crack tip in an elastic perfectly-plastic material[1]. For the steady-state case it was found that the dynamic near-tip fields can be expressed as simple-wave solutions of the governing system of hyperbolic partial differential equations. These solutions are independent of the dimensionless distance to the crack tip, r/r_p, but they do depend specifically on the dimensionless crack-tip speed M. For Mode-III crack propagation the simple-wave solution in the near-tip field is a combination of a centered-fan field and a uniform field. Explicit expressions have been presented in Refs.[1] and [2].

The solutions that were obtained in Ref.[1] show some anomalies in the transition from the dynamic to the quasi-static solution. As the crack-tip speed, M, decreases the expressions for the stresses reduce to the ones for the corresponding quasi-static solution, as might be expected on the basis of intuitive reasoning. This is however not true for the strains, which become unbounded in the limit of vanishing M. In Ref.[1] it was speculated that the transition from dynamic to quasi-static conditions with decreasing crack-tip speed is effected because the dynamic solution is asymptotically valid in a small edge zone, which shrinks on the crack tip in the limit of vanishing crack-tip speed.

In this paper this non-uniform transition has been investigated in detail in the plane of the crack for the case of crack propagation in anti-plane strain (Mode-III). It is shown that for small crack-tip speeds the complete near-tip solution consists of the outer solution, which is a regular perturbation expansion in M, with the quasi-static solution as the first term, and the inner solution, which is of completely different nature with a strong influence of dynamic effects. The inner solution is valid in an edge-zone which is analogous to a boundary layer. The first term in a uniform expansion over the polar angle θ measured from the plane of the crack displays the connection between the inner and outer solutions.

The governing equations and the boundary conditions are stated in Section 2. The outer solution, its relation to the quasi-static solution, as well as its inadequacy in the immediate vicinity of the crack tip have been discussed in Section 3. The results for the inner solution have been summarized in Section 4. It is shown in Section 5, that certain functions which define the fields for small polar angle from the plane of the crack, satisfy a system of coupled ordinary differential equations. An investigation of the critical points in the phase plane and an inspection of the trajectories of the solutions of these equations as the distance to the crack tip decreases, reveals that the expressions of Section 4 do indeed provide the solution in an edge zone. Exact solutions in implicit form for the coupled ordinary differential equations mentioned above have also been obtained in Section 5, and these solutions reproduce the inner solution as well as the outer solution.

Finally, the system of ordinary differential equations has been solved numerically, to yield exact solutions for the shear strains in the zone of plastic deformation ahead of the crack tip, in and near the plane of the crack. The results, which have been plotted vs the distance from the crack tip, for various values of the crack-tip speed, show the influence of dynamic effects.

2. GOVERNING EQUATIONS

A coodinate system (x, y, z) is attached to the moving crack tip as shown in Fig. 1. The assumption that a steady-state has been established relative to the moving crack tip, implies that absolute time derivatives take the following forms relative to the moving coordinate system:

$$\partial_t(\ ) = -v\partial_x(\ ); \quad \partial_{tt}(\ ) = v^2\partial_{xx}(\ ). \tag{2.1a,b}$$

In the moving coordinate system steady-state deformation in anti-plane strain is defined by a displacement in the z-direction, which is denoted by $w(x, y)$. The corresponding stress components are $\sigma_{xz}(x, y)$ and $\sigma_{yz}(x, y)$.

In view of (2.1b) the equation of motion becomes

$$\partial_x\sigma_{xz} + \partial_y\sigma_{yz} = \rho v^2\partial_{xx}w. \tag{2.2}$$

The Mises yield condition is

$$\sigma_{xz}^2 + \sigma_{yz}^2 = k^2, \tag{2.3}$$

where k is the yield stress in shear. By virtue of (2.1a), the Prandtl–Reuss equations for an elastic perfectly-plastic solid reduce to the forms

$$\partial_{xx}w = \mu^{-1}\partial_x\sigma_{xz} - \Lambda\sigma_{xz} \tag{2.4}$$

$$\partial_{xy}w = \mu^{-1}\partial_x\sigma_{yz} - \Lambda\sigma_{yz}, \ \Lambda \geq 0. \tag{2.5}$$

Here μ is the elastic shear modulus, and Λ is a non-negative proportionality factor, which may vary in space.

In the region of plastic deformation the yield condition is identically satisfied by

$$\partial_{xy}w = -k\sin\omega, \quad \sigma_{yz} = k\cos\omega. \tag{2.6a,b}$$

Elimination of Λ from (2.4) and (2.5), and the use of (2.2) and (2.6a,b) then yields the following system of equations

$$\cos\omega\partial_x\omega + \sin\omega\partial_y\omega + M^2\frac{\mu}{k}\partial_x\gamma_x = 0 \tag{2.7}$$

$$\cos\omega\partial_x\gamma_x + \sin\omega\partial_y\gamma_x + \frac{k}{\mu}\partial_x\omega = 0 \tag{2.8}$$

where

$$\gamma_x = \partial_x w \tag{2.9}$$

$$M = v/(\mu/\rho)^{1/2}. \tag{2.10}$$

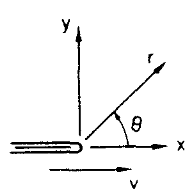

Fig. 1. Propagating crack tip (velocity v) with moving coordinate system.

The condition of vanishing σ_{yz} on the crack faces yields by virtue of (2.6b)

$$y = 0, x < 0: \ \omega = \pi/2. \tag{2.11}$$

Displacement anti-symmetry relative to $y = 0$ implies that $w \equiv 0$ for $y = 0$, $x > 0$. By virtue of (2.6a) we then have

$$y = 0, x > 0: \ \omega = 0. \tag{2.12}$$

The strain component γ_y, where

$$\gamma_y = \partial_y w, \tag{2.13}$$

does not enter in the governing eqns (2.7) and (2.8). Once γ_x has been obtained, we can, however, compute γ_y from the relation

$$\partial_x \gamma_y = \partial_y \gamma_x. \tag{2.14}$$

In the polar coordinate system shown in Fig. 1, (2.7) and (2.8) have the forms

$$\cos(\omega - \theta)\partial_r\omega + \frac{1}{r}\sin(\omega - \theta)\partial_\theta\omega + \frac{\mu M^2}{k}\left(\cos\theta\partial_r\gamma_x - \frac{1}{r}\sin\theta\partial_\theta\gamma_x\right) = 0 \tag{2.15}$$

$$\cos(\omega - \theta)\partial_r\gamma_x + \frac{1}{r}\sin(\omega - \theta)\partial_\theta\gamma_x + \frac{k}{\mu}\left(\cos\theta\partial_r\omega - \frac{1}{r}\sin\theta\partial_\theta\omega\right) = 0. \tag{2.16}$$

At the elastic-plastic boundary the displacement is continuous, and hence $\partial w/\partial s$ is continuous, where s is a variable along the elastic-plastic boundary. This condition may be expressed in the form

$$\left[\frac{\partial w}{\partial s}\right] = -\left[\frac{\partial w}{\partial x}\right]n_y + \left[\frac{\partial w}{\partial y}\right]n_x = 0. \tag{2.17}$$

Here [ ] denotes a discontinuity across the elastic-plastic boundary, and n_x, n_y are the components of the outer normal $\mathbf{n}$.

A second condition follows from the impulse-momentum relation at the elastic-plastic boundary, which states $[\sigma_{ij}]n_j = -\rho c[\dot{u}_i]$, where c is the velocity in the direction of $\mathbf{n}$. For the present steady-state case we find

$$[\sigma_{xz}]n_x + [\sigma_{yz}]n_y = \mu M^2[\gamma_x]. \tag{2.18}$$

Let us define

$$(\gamma_{y0}^{\pm}, \sigma_{yz0}^{\pm}) = \lim_{y=0,\, x\to r_p\pm 0} (\gamma_y, \sigma_{yz}) \tag{2.19}$$

where $x = r_p$ defines the elastic-plastic boundary for $y = 0$, $x > 0$. For $y = 0$, $x \geq 0$, we have $w \equiv 0$. It then follows from (2.17) that

$$\gamma_{y0}^{-} = \gamma_{y0}^{+}. \tag{2.20}$$

Since the yield condition (2.3) holds at the elastic-plastic boundary, and since $\sigma_{xz} \equiv 0$ for $y = 0$, $x > 0$, we have

$$\sigma_{yz0}^{-} = \sigma_{yz0}^{+} = k. \tag{2.21}$$

From $\gamma_{y0}^{+} = \sigma_{yz0}^{+}/\mu$, and (2.20)–(2.21) we subsequently obtain

$$\gamma_{y0}^{-} = k/\mu. \tag{2.22}$$

3. FAR-FIELD SOLUTION

Let us consider regular series expansions for ω and γ_x with respect to the "Mach number" M defined by (2.10)

$$\omega = \bar{\omega}_0 + M\bar{\omega}_1 + \cdots \tag{3.1}$$

$$\gamma_x = \bar{\gamma}_{x0} + M\bar{\gamma}_{x1} + \cdots \tag{3.2}$$

For the leading terms $\bar{\omega}_0$ and $\bar{\gamma}_{x0}$ we obtain the following equations

$$\cos(\bar{\omega}_0 - \theta)\partial_r\bar{\omega}_0 + \frac{1}{r}\sin(\bar{\omega}_0 - \theta)\partial_\theta\bar{\omega}_0 = 0 \tag{3.3}$$

$$\cos(\bar{\omega}_0 - \theta)\partial_r\bar{\gamma}_{x0} + \frac{1}{r}\sin(\bar{\omega}_0 - \theta)\partial_\theta\bar{\gamma}_{x0} + \frac{k}{\mu}\left(\cos\theta\partial_r\bar{\omega}_0 - \frac{1}{r}\sin\theta\partial_\theta\bar{\omega}_0\right) = 0. \tag{3.4}$$

These equations govern the quasi-static problem, which was discussed by Chitaley and McClintock[3]. For small values of r/r_p (where r_p defines the boundary of the plastic domain) we obtain (see Ref. 3)

$$\bar{\omega}_0 = \theta \tag{3.5}$$

$$\bar{\gamma}_{x0} = \frac{k}{\mu}\sin\theta\ln\left(\frac{r}{r_p}\right) + f(\theta) \tag{3.6}$$

where $f(\theta)$ is an arbitrary function of θ. In the plane of the crack, the first term in a corresponding expansion for γ_y follows from (2.14) and (3.6) as

$$\bar{\gamma}_{y0} = \frac{k}{\mu}\left[\frac{1}{2}\ln^2\left(\frac{r}{r_p}\right) - \ln\left(\frac{r}{r_p}\right) + 1\right] \tag{3.7}$$

where the condition (2.22) has been used.

It is of interest to examine the magnitude of the inertia term, $M^2\partial_x^2 w = M^2\partial_x\gamma_x$, which would correspond to the quasi-static solution. By the use of (3.5) and (3.6) we find

$$M^2\partial_x\gamma_x \approx \frac{k}{2\mu}\frac{M^2}{r}\left[1 - \ln\left(\frac{r}{r_p}\right) - \frac{f'(\theta)}{\cos\theta}\right]\sin 2\theta. \tag{3.8}$$

On the other hand, if the stress derivatives appearing in eqn (2.2) are computed on the basis of (3.5) and (3.6), it is found that they vanish identically. Equation (3.8) then suggests that the quasi-static approximation is not valid in a small neighborhood of the crack-tip defined by $r/r_p \sim 0(M^2)$ (we shall make a more accurate estimate in a later Section). The subsequent terms in the series (3.1) and (3.2) do not remove this nonuniformity of the approximation. Thus the regular expansion with respect to M cannot be accepted in the immediate vicinity of the crack-tip. It appears that this solution represents an outer or "far-field" expansion.

4. NEAR-FIELD SOLUTION

In a recent paper Achenbach and Dunayevsky[1] have shown that the solution (ω, γ_x) to eqns (2.7) and (2.8) is either singular near the crack tip, or it is represented by a centered fan-field in combination with a uniform field. Since it was not possible to obtain a singular solution, the centered fan + uniform field solution was considered in some detail. The solution was obtained in the following form[1, see also 2]:

$0 \le \theta \le \theta^*$

$$\gamma_x = -\frac{k}{\mu M}\cos^{-1}[M\sin^2\theta + (1 - M^2\sin\theta)^{1/2}\cos\theta] \tag{4.1}$$

$$\sigma_{xz} = -k[(1 - M^2\sin^2\theta)^{1/2} - M\cos\theta]\sin\theta \tag{4.2}$$

$$\sigma_{yz} = k[(1 - M^2\sin^2\theta)^{1/2}\cos\theta + M\sin^2\theta] \tag{4.3}$$

$\theta^* \le \theta \le \pi$

$$\sigma_{xz} = -k,\ \sigma_{yz} = 0,\ \gamma_x = -k\pi/2\mu M \tag{4.4a,b}$$

where

$$\theta^* = \tan^{-1}(1/M). \tag{4.5}$$

The corresponding strains γ_y follow from (2.14) as:

$0 \leq \theta \leq \theta^*$

$$\gamma_y = \frac{k}{\mu}\left\{\frac{1-M}{2M}\ln\left[1 - M\sin^2\theta - (1-M^2\sin^2\theta)^{1/2}\cos\theta\right] + \frac{1+M}{2M}\ln\left[1 + M^2\sin^2\theta + (1-M^2\sin^2\theta)^{1/2}\cos\theta\right]\right\} + \psi(y) \tag{4.6}$$

$\theta^* \leq \theta \leq \pi$

$$\gamma_y = \psi(y). \tag{4.7}$$

Here $\psi(y)$ is an unknown function.

In the limit $M \to 0$ the stress fields reduce to the corresponding quasistatic fields. This is, however, not true for the strains, which become unbounded as $M \to 0$. Difficulties of this kind in the transition from the dynamic to the quasi-static solution as $M \to 0$ could have been anticipated from the structure of the governing equations. As $M \to 0$ the two distinct families of characteristic curves of eqns (2.6) and (2.7), which are defined by

$$\frac{dx}{dy} = \cot\omega \pm \frac{M}{\sin\omega} \tag{4.8}$$

merge into one family, defined by

$$\frac{dx}{dy} = \cot\omega. \tag{4.9}$$

Such a degeneracy usually leads to a non-uniform transition, and the appearance of a "boundary layer". Some examples are given by Cole[4].

Another indication of a non-uniform transition is given by the form of the equations in the hodograph plane. To transform (2.7) and (2.8) to the hodograph plane we introduce the following changes of variables[5].

$$\frac{\partial x}{\partial \omega} = J^{-1}\frac{\partial \gamma_x}{\partial y}, \quad \frac{\partial y}{\partial \omega} = -J^{-1}\frac{\partial \gamma_x}{\partial x} \tag{4.10a,b}$$

$$\frac{\partial x}{\partial w_x} = -J^{-1}\frac{\partial \omega}{\partial y}, \quad \frac{\partial y}{\partial w_x} = J^{-1}\frac{\partial \omega}{\partial x} \tag{4.11a,b}$$

where J is the Jacobian

$$J = \frac{\partial \omega}{\partial x}\frac{\partial \gamma_x}{\partial y} - \frac{\partial \gamma_x}{\partial x}\frac{\partial \omega}{\partial y}. \tag{4.12}$$

In the hodograph plane eqns (2.7) and (2.8) take the form

$$\cos\omega\frac{\partial y}{\partial \gamma_x} - \sin\omega\frac{\partial x}{\partial \gamma_x} - M^2\frac{k}{\mu}\frac{\partial y}{\partial \omega} = 0 \tag{4.13}$$

$$-\cos\omega\frac{\partial y}{\partial \omega} + \sin\omega\frac{\partial x}{\partial \omega} + \frac{k}{\mu}\frac{\partial y}{\partial \gamma_x} = 0. \tag{4.14}$$

This linear system of equations can be reduced to a single equation of the second order

$$\left(\frac{k}{\mu}\right)^2\frac{\partial^2 y}{\partial \gamma_x^2} - M^2\frac{\partial^2 y}{\partial \omega^2} + M^2\cot\omega\frac{\partial y}{\partial \omega} - \frac{1}{\sin\omega}\frac{\partial y}{\partial \gamma_x} = 0. \tag{4.15}$$

when $M \to 0$ the second order derivative $\partial^2 y/\partial\omega^2$ disappears, which indicates that an asymptotic expansion with respect to M cannot be uniform.

On the basis of the foregoing observations it may be assumed that the field near the crack tip is of different forms in two zones. The outer solution in the "far field" is a regular expansion in M as given by (3.1) and (3.2), where the leading term represents the quasi-static solution. The effect of inertia is relatively small in this far-field. Inertia is, however, important in the inner solution in the near field, represented by (4.1)–(4.5). The near-field can be thought of as a "boundary layer". In the present geometry the terminology "edge zone" is, however, more appropriate. In the edge-zone the inertia effects appear to remove the singularity of the quasi-static strain component γ_y. The strain component γ_y remains, however, singular.

In the next section we consider the matching of the inner solution and the outer solution, and the shrinking of the edge-zone on the crack tip as $M \to 0$.

5. SOLUTION IN THE PLANE OF THE CRACK

Matching of the far-field and near-field solutions has not been accomplished for arbitrary values of the polar angle θ. It is, however, possible to investigate the matching for small values of θ, by seeking a solution in the following form

$$\omega = \omega_1(r)\theta + \omega_3(r)\theta^3 + \cdots \tag{5.1}$$

$$\gamma_x = \gamma_{x1}(r)\theta + \gamma_{x3}(r)\theta^3 + \cdots \tag{5.2}$$

Substitution of (5.1) into (2.15) and (2.16) and collecting the terms of equal powers of θ, gives for the first approximation the following equations

$$\frac{d\omega_1}{dr} + \frac{\omega_1(\omega_1 - 1)}{r} + \frac{\mu M^2}{k}\left(\frac{d\gamma_{x1}}{dr} - \frac{\gamma_{x1}}{r}\right) = 0 \tag{5.3}$$

$$\frac{d\gamma_{x1}}{dr} + \frac{\gamma_{x1}(\omega_1 - 1)}{r} + \frac{k}{\mu}\left(\frac{d\omega_1}{dr} - \frac{\omega_1}{r}\right) = 0. \tag{5.4}$$

At this stage it is useful to introduce new variables

$$J_+ = M\gamma_{x1} + \frac{k}{\mu}\omega_1, \quad J_- = M\gamma_{x1} - \frac{k}{\mu}\omega_1. \tag{5.5a,b}$$

Inserting (5.5) into (5.3) and (5.4) we arrive at

$$(1+M)\frac{dJ_+}{d\alpha} + (1+M)J_+ - \frac{\mu}{2k}(J_+ - J_-)J_+ = 0 \tag{5.6}$$

$$(1-M)\frac{dJ_-}{d\alpha} + (1-M)J_- - \frac{\mu}{2k}(J_+ - J_-)J_- = 0 \tag{5.7}$$

where the following change of variables has been used

$$\alpha = -\ln(r/r_p). \tag{5.8}$$

Let us consider the solutions to eqns (5.6) and (5.7) in the phase plane (J_+, J_-). The critical points are the solutions to the following equations

$$(1+M)J_+ - \frac{\mu}{2k}(J_+ - J_-)J_+ = 0 \tag{5.9}$$

$$(1-M)J_- - \frac{\mu}{2k}(J_+ - J_-)J_- = 0. \tag{5.10}$$

Equations (5.9) and (5.10) define three critical points, whose position and character are given by by

(1) $J_+ = J_- = 0$; stable node (5.11)

(2) $J_- = 0$; $J_+ = (1+M)\dfrac{2k}{\mu}$; unstable node (5.12)

(3) $J_+ = 0$; $J_- = -(1-M)\dfrac{2k}{\mu}$; saddle point. (5.13)

The nature of the critical points given by (5.11)–(5.13) determines the phase flow pattern, i.e. the trajectories of the solutions as α varies, shown in Fig. 2 (see e.g. Ref. [6]).

Since no limit cycle or center point exists, the trajectories beneath the separatrix line AA' run towards infinity as $\alpha \to \infty$ ($r \to 0$). Along these lines J_+ and J_-, and consequently ω_1 and γ_{x1}, become unbounded. However infinite growth of ω_1 leads to an oscillation of the stress field (with increasing frequency) which is not admissible from the physical point of view. Thus this domain of the phase plane falls outside our consideration.

The phase flow above the separatrix line AA' tends to the origin $J_+ = J_- = 0$ so that for $\alpha \to \infty$ ($r \to 0$) we have $J_+ \to 0$, $J_- \to 0$, and consequently $\omega_1 \to 0$, $\gamma_{x1} \to 0$. This result would correspond to a uniform zero field ahead of the crack tip. The hyperbolicity of the governing eqns (2.7) and (2.8) then would imply a zero field for the whole loading domain, which does, however, not satisfy the boundary condition on the crack faces and therefore has to be rejected.

Thus we have no other alternative but the trajectory along the separatrix AA'. There is however a "forbidden" region in the phase plane which is defined by the condition $\Lambda \geq 0$. It follows from (2.4), (2.5) and (5.1), (5.2) that

$$k\Lambda \approx -\frac{\gamma_{x1}}{r}. \tag{5.14}$$

The requirement $\Lambda \geq 0$ leads to the inequality

$$-\gamma_{x1} = -\frac{1}{2M}(J_+ + J_-) \geq 0 \tag{5.15}$$

which implies that we have to consider the part of the separatrix which lies on the left of the bisector $J_+ = -J_-$.

Thus, when $\alpha \to \infty$ ($r \to 0$) the solution moves along the line AA' toward the point 3. Let

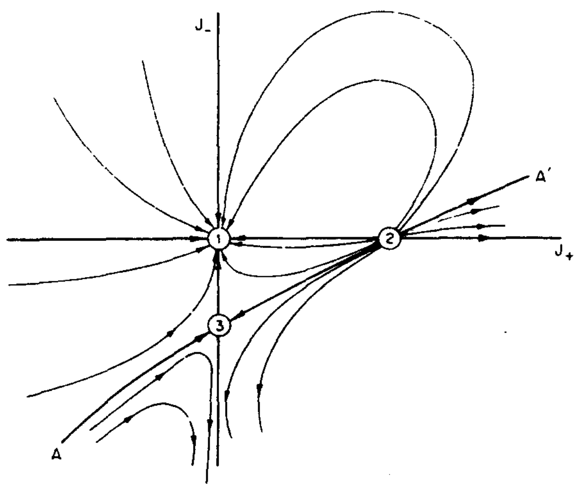

Fig. 2. Phase plane with critical points, for eqns (5.6) and (5.7).

us now find the solution in the neighborhood of the point 3. Linearization of eqns (5.6) and (5.7) near this point leads to the following equations

$$\frac{dJ_+}{d\alpha} = -\frac{2M}{1+M} J_+ \tag{5.16}$$

$$\frac{dJ_-}{d\alpha} = -J_+ + J_- + (1-M)\frac{2k}{\mu}. \tag{5.17}$$

In terms of the variable r the solution has the form

$$J_+ = C\frac{1+3M}{2M}(r/r_p)^{2M/(1+M)} \tag{5.18}$$

$$J_- = -(1-M)\frac{2k}{\mu} + C(r/r_p)^{2M/(1+M)} \tag{5.19}$$

where C is an arbitrary constant and (5.8) has been used. Hence by virtue of (5.5) we obtain

$$\gamma_{x1} = -\frac{k}{\mu M}(1-M) + C\frac{1+5M}{4M^2}(r/r_p)^{2M/(1+M)} \tag{5.20}$$

$$\omega_1 = 1 - M + C\frac{\mu}{2k}\frac{1+M}{2M}(r/r_p)^{2M/(1+M)}. \tag{5.21}$$

Hence we have

$$\omega \approx \omega_1\theta = (1-M)\theta + 0[(r/r_p)^{2M/(1+M)}] \tag{5.22}$$

$$\gamma_x \approx \gamma_{x1}\theta = -\frac{k}{\mu M}(1-M)\theta + 0[(r/r_p)^{2M/(1+M)}] \tag{5.23}$$

The dominant terms of the expressions (5.22) and (5.23) coincide with those of the expansion of the near field, (4.1) and (4.4), for small θ. This result confirms the validity of the simple wave solution near the crack tip.

Equations (5.6) and (5.7) can be integrated to yield the following solution in implicit form

$$J_+ = A(r/r_p)^{2M/(1+M)}|J_-|^{(1+M)/(1+M)} \tag{5.24}$$

$$F\left(1, \frac{1+M}{2M}, 1+\frac{1+M}{2M}, -A\left(\frac{r}{r_p}\right)^{2M/(1+M)}|J_-|^{-2M/(1+M)}\right) = -\frac{1}{1-M}\frac{\mu}{2k}J_- + BJ_-\left(\frac{r}{r_p}\right)^{-1}. \tag{5.25}$$

Here A and B are constants of integration and $F(p, q, r, s)$ is the hypergeometrical function. The trajectory along the separatrix AA' corresponds to $B = 0$. This follows from the observation that the separatrix crosses the line $J_+ = 0$ at the point 3, which is defined by (5.13).

It can now be verified that for $(r/r_p) \to 0$, (5.24) and (5.25) yield $J_+ \to 0$ and $J_- \to (2k/\mu)(1-M)$, which correspond to $\omega_1 \to 1-M$ and $\gamma_{x1} \to -(k/\mu M)(1-M)$. These results agree with (5.20) and (5.21).

Next we consider (5.24) and (5.25) in the limit $M \to 0$. It is noted from (5.5) that for $M = 0$ we have $J_+ = -J_-$. Since $J_- < 0$, the constant A in (5.24) must equal unity: $A = 1$. For $M \neq 0$, the combination of (5.5a,b) and (5.24) then yields

$$\ln\left(M\gamma_{x1} + \frac{k}{\mu}\omega\right) = \frac{2M}{1+M}\ln\left(\frac{r}{r_p}\right) + \frac{1-M}{1+M}\ln\left|M\gamma_{x1} - \frac{k}{\mu}\omega\right| + \ln A. \tag{5.26}$$

To expand (5.26) for small M we require $r/r_p > \delta$ where δ is such that

$$-\ln\left(\frac{r}{r_p}\right) \sim 0\left(\frac{1}{M}\right). \tag{5.27}$$

Equation (5.26) then yields

$$\gamma_{x1} = \frac{k}{\mu}\,\omega\left[\ln\left(\frac{r}{r_p}\right) - \ln\left(\frac{k}{\mu}\,\omega\right) + \frac{1}{2M}\ln A\right] + 0(M). \tag{5.28}$$

In deriving this expression, it was taken into account that $J_- < 0$.

For the hypergeometrical function in eqn (5.25) we find for small M

$$F\left(1, \frac{1+M}{2M}, 1 + \frac{1+M}{2M}, -\left(\frac{r}{r_p}\right)^{2M/(1+M)} |J_-|^{-2M/(1+M)}\right)$$
$$\approx F\left(1, \frac{1+M}{2M}, 1 + \frac{1+M}{2M}, -1\right) + 0(M^2) = \frac{1+M}{2M}\,\beta\left(\frac{1+M}{2M}\right) + 0(M^2). \tag{5.29}$$

Here $\beta(x)$ is the β-function

$$\beta(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{n!}{x(x+1)\cdots(x+n)}\frac{1}{2^n}. \tag{5.30}$$

By virtue of (5.29), (5.30) and (5.25) we obtain for $M \to 0$

$$\frac{1}{2} + 0(M) = -(1+M)\frac{\mu}{2k}\left(M\gamma_{x1} - \frac{k}{\mu}\,\omega\right) \tag{5.31}$$

For $(r/r_p) > \delta$ the formulas (5.26) and (5.31) imply

$$\omega_1 \sim 1,\ \gamma_{x1} \sim \frac{k}{\mu}\ln\left(\frac{r}{r_p}\right) + \ln(\text{const.}),\ A = (\text{const.})^{2M}. \tag{5.32}$$

This result agrees with the quasi-static given by (3.5)–(3.6), which coincides with the dominant terms of the outer solution for small θ, as given by (3.1)–(3.2).

In summary, for small values of the angle θ the solution given by (5.24) and (5.25) matches the near field (4.1) and (4.3) to the far field (3.5)–(3.6) for small values of the angle θ. The region defined by $\ln(r/r_p) \sim 0(M^{-1})$ can be identified as an edge zone.

To follow in some detail the phase flow when $M \to 0$ it is convenient to consider the (γ_x, ω) phase plane. The point of intersection of the separatrix AA' with the ω-axis should be located between $1 - M$ and $1 + M$. The angle χ of intersection of the J_+ and J_- lines, is given by

$$\chi = \pi - 2\tan^{-1}(1/M),$$

which tends to zero when $M \to 0$. The abscissa of the saddle point 3 is $\omega_3 = 1 - M$ and the abscissa of the node 2 is $\omega_2 = 1 + M$, which both tend to 1 as $M \to 0$. Thus the critical points 2 and 3 shift to infinity as $M \to 0$. For $M \equiv 0$ the phase flow has degenerated in that the separatrix has turned into a straightline which is parallel to the γ_x-axis and has the abscissa $\omega = 1$. The magnitude γ_{x1} has become unbounded along this line.

From the point of view of fracture mechanics it is of interest to compute the strain $\gamma_y = \partial w/\partial y$ in the plane of the crack. Analogously to (5.1) and (5.2) we may consider the expansion

$$\gamma_y = \gamma_{y0}(r) + \gamma_{y2}(r)\theta^2. \tag{5.33}$$

Equation (2.14) then yields

$$\mathrm{d}\gamma_{y0}/\mathrm{d}r = \gamma_{x1}/r. \tag{5.34}$$

By employing (5.5a,b) to express γ_{x1} in terms of J_+ and J_-, we obtain from (5.34)

$$\gamma_{y0} = \frac{1}{2M} \int r^{-1}(J_+ + J_-)\,dr. \tag{5.35}$$

To evaluate (5.35) we first substitute (5.24) into (5.6), and we reintroduce r by the use of (5.8) to obtain

$$-r\frac{dJ_+}{dr} + J_+ - \frac{\mu}{2k}\frac{1}{1+M}[J_+^2 - (J_+/A)^{(1+M)/(1-M)}(r/r_p)^{-2M/(1-M)}] = 0. \tag{5.36}$$

By defining the new variable I_+ by

$$J_+ = \bar{r}I_+, \text{ where } \bar{r} = r/r_p, \tag{5.37}$$

Equation (5.36) can be rewritten as

$$d\bar{r} = -\frac{2k}{\mu}(1+M)\frac{dI_+}{I_+^2 - A^{-(1+M)/(1-M)}I_+^{2M/(1-M)}} \tag{5.38}$$

Substitution of (5.24) into (5.35), and subsequent substitution of (5.37) and (5.38) in the resulting integral yields

$$\gamma_{yo} = -\frac{k}{\mu}\frac{1-M}{M}\int \frac{1 + A^{-(1+M)/(1-M)}I_+{}^{2M/(1-M)}}{I_+[1 - A^{-(1+M)/(1-M)}I_+^{2M/(1-M)}]}\,dI_+. \tag{5.39}$$

This integral can be evaluated to yield the following result

$$\frac{\mu}{k}\gamma_{y0} = D - \frac{1-M^2}{M^2}\left\{\ln 2 + \frac{1}{2}\frac{1+M}{1-M}\ln A + \ln\left|\left(\frac{J_+}{\bar{r}}\right)^{2M/(1-M)} - A^{(1+M)/(1-M)}\right| - \frac{M}{1-M}\ln\left(\frac{J_+}{\bar{r}}\right)\right\} \tag{5.40}$$

where D is a constant. The constant D can be obtained by applying the condition (2.22) at $\bar{r} = 1$. The result is

$$\frac{\mu}{k}\gamma_{y0} = 1 + \frac{1-M^2}{M^2}\ln\left|\frac{(J_+/\bar{r})^{2M/(1-M)} - A^{(1+M)/(1-M)}}{(J_+^*)^{2M/(1-M)} - A^{(1+M)/(1-M)}}\right| + \frac{1+M}{M}\ln|J_+/J_+^*\bar{r}| \tag{5.41}$$

where

$$J_+^* = J_+|_{\bar{r}=1}. \tag{5.42}$$

For $\bar{r} = r/r_p \to 0$, J_+ is given by (5.18). Substitution of this result in (5.41) gives

$$\frac{\mu}{k}\gamma_{y0} = -\frac{1-M}{M}\ln\left(\frac{r}{r_p}\right) + 0(1). \tag{5.43}$$

By using (5.32), we find for $M \to 0$ outside of the edge zone

$$\frac{\mu}{k}\gamma_{y0} = \frac{1}{2}\ln^2\left(\frac{r}{r_p}\right) - \ln\left(\frac{r}{r_p}\right) + 1 + 0(M). \tag{5.44}$$

This expression agrees with (3.7).

6. FURTHER EVALUATION OF EXACT SOLUTIONS

Equations (5.24) and (5.25), together with (5.5a,b) and (5.41) provide exact solutions for ω_1, γ_{x1} and γ_{y0}. Equations (5.24) and (5.25) still contain, however, one as yet unknown constant A.

This constant can be determined by a further consideration of the condition on the elastic–plastic boundary given by eqn (2.18). Upon substitution of the expansion (5.1) into (2.6a,b), and the subsequent substitution of the result as well as (5.2) into (2.18), together with

$$\sigma^+_{xz} = \sigma^+_{xz1}\theta + O(\theta^3), \tag{6.1}$$

where $\sigma^+_{xz1} = \mu\gamma^+_{x1}$, we find at $r = r_p$

$$M^2\gamma^-_{x1} + \frac{k}{\mu}\omega_1 = -\frac{1}{\mu}(1 - M^2)\sigma^+_{xz1}. \tag{6.2}$$

The $\pm$ notation has been defined by (2.19). By the use of (5.5a,b) this condition can be rewritten in the form

$$(1+M)J^*_+ - (1-M)J^*_- = -\frac{2}{\mu}(1 - M^2)\sigma^*_{xz1}. \tag{6.3}$$

Two other boundary conditions follow from (5.24) and (5.25) as

$$J^*_+ = A|J^*_-|^{(1-M)/(1+M)} \tag{6.4}$$

$$F\left(1, \frac{1+M}{2M}, 1 + \frac{1+M}{2M}, -A|J^*_-|^{-2M/(1+M)}\right) = -\frac{1}{1-M}\frac{\mu}{2k}J^*_-. \tag{6.5}$$

Thus J^*_+, J^*_- and A can be solved from (6.3)–(6.5), in terms of the elastic stress component σ^+_{xz1}. Once A is known, eqns (5.24) and (5.25) can be solved for J_+ and J_-.

The problem simplifies considerably if it is assumed that $[\gamma_{x1}] \equiv 0$. Equation (6.2) then yields

$$(k/\mu)\omega_1 = -\gamma_{x1} \text{ at } r = r_p, \text{ which gives the condition } J^*_+ = -\frac{1-M}{1+M}J^*_-. \tag{6.6}$$

It is noted that the elastic field no longer appears in (6.4)–(6.6), except implicitly in the length of the plastic zone r_p.

As an alternative to solving the implicit forms (5.24) and (5.25), eqns (5.6) and (5.7) may be solved numerically. This has actually been done in this paper, with the simplified condition (6.6). The function $\gamma_{x1} = (J_+ + J_-)/2M$ has been plotted in Fig. 3. The corresponding quasi-static solution, which is indicated by $M = 0$, has also been plotted. The quasi-static solution for γ_{x1} is singular at $r/r_p = 0$, while the dynamic solution remains bounded. The curve for $M = 0.01$ is very close to the one for $M \equiv 0$, i.e. to $\ln(r/r_p)$, in the region $r/r_p > \delta$, where δ is the "length" of

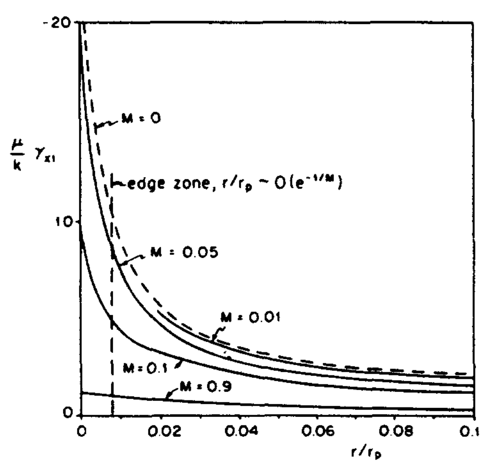

Fig. 3.

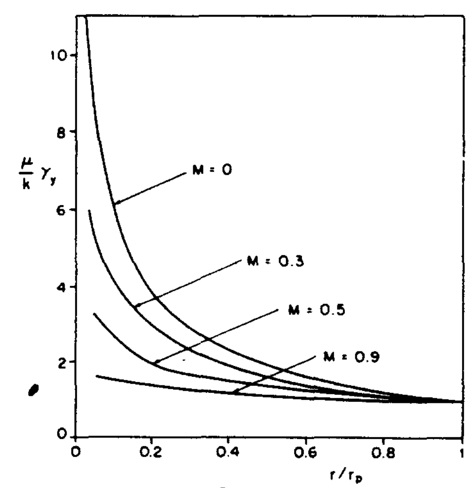

Fig. 4.

Fig. 3. Shear strain, $(\mu/k)\gamma_{x1}$ vs r/r_p for various crack tip speeds, with an estimate of the edge zone.

Fig. 4. Shear strain γ_y in the plane of the crack ($y = 0$, $x > 0$) vs r/r_p for various crack-tip speeds.

the edge zone in the plane of the crack. An estimate of the length of the edge zone is indicated in Fig. 3.

Curves for γ_{y0} have been computed by substituting the numerical solution for γ_{x1} in (5.34) and integrating numerically, using the condition (2.22) at $r = r_p$. The curves have been plotted in Fig. 4. Substantial differences are noted between the quasi-static ($M = 0$) and the dynamic shear strains.

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